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Strong-interaction approximation for transfer matrix method

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Abstract Using transfer-matrix method a correspondence between $2D$ classical spin systems ($2D$ Ising model and six-vertex model) and $1D$ quantum spin systems is considered. We find the transfer matrix in two limits - in a well-known strong-anisotropy limit and a novel strong-interaction limit. In contrast to the usual strong-anisotropy approximation, within the strong-interaction approximation we take into account the non-commutativity of transfer-matrix components. The latter approximation is valid for low temperatures or strong interaction in one spatial dimension. We observe that the Hamiltonian of the corresponding quantum chains contains multispin interactions.

Keywords transfer matrix · quantum spin chains

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1 Introduction

Transfer matrix method allows to find thermodynamic properties for many low-dimensional models [1], [2] and shows thermodynamic equivalence for many of them [3], [4]. Important feature of the method is the possibility to establish a relation between thermodynamic of d -dimensional classical models and ground state of $(d - 1)$ -dimensional quantum models [5] under the assumption of strongly anisotropic interactions in a classical system.

In the present paper we discuss the conditions which are imposed on a classical system within strong-anisotropy approximation. Besides that, a new strong-interaction approximation which requires weaker conditions is introduced. For the new approximation the transfer matrix is written in the symmetric form $T =$

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$e^{t_x X} e^{t_y Y} e^{t_x X}$, see (9), and the expansion in nested commutators similar to the Baker-Campbell-Hausdorff formula is performed. The quantum Hamiltonian can be found explicitly only if the nested commutator of an arbitrary order $[X, \dots [X, Y] \dots]$ can be written in a specific form, (see A10). The strong-interaction limit reduces to the strong-anisotropy limit if we assume $[X, Y] = 0$. The strong-anisotropy limit also can be obtained by applying the exponential operator decomposition technique [6], [7]. The main difference between the operator decomposition technique and the strong-interaction limit is that in the latter case we must calculate nested commutators instead of assuming that $t_x \propto t_y$. For the $2D$ spin- $\frac{1}{2}$ classical Ising model the strong-interaction limit leads to the appearance of three-spin interactions in the resulting quantum chain. For the six-vertex model the new approximation gives the XXZ chain with four-spin interactions.

We will use the transfer matrix method for classical $2D$ models [8] with the total energy that can be represented as a sum over rows

$$E = \sum_{m=1}^M E(\xi_m, \xi_{m+1}), \quad (1)$$

where ξ_m is a variable defined on the row m with \mathbb{L} possible values. The free energy of the classical $2D$ model (per row) can be written in the form

$$f = - \lim_{M \rightarrow \infty} \frac{1}{\beta M} \ln \text{Tr} T^M = - \frac{1}{\beta} \ln \lambda_{\max}, \quad (2)$$

where T is the transfer matrix with elements $T_{\xi_m, \xi_{m+1}} = e^{-\beta E(\xi_m, \xi_{m+1})}$ and the maximal eigenvalue λ_{\max} which is real, unique and positive according to the Perron-Frobenius theorem [9] for the positive symmetric matrix T . We can consider \mathbb{L} configurations as orthonormal basis $\{|\xi\rangle\}_{\xi \in \mathbb{L}}$ of \mathbb{L} -dimensional Hilbert space. Each configuration ξ corresponds to a base-vector in the Hilbert space. The transfer matrix T corresponds to some operator $\mathbf{T} = \sum_{\xi, \xi'=1}^{\mathbb{L}} |\xi\rangle T_{\xi, \xi'} \langle \xi'|$. Let us introduce the Hamiltonian of a quantum system, which is defined by the logarithm of the operator \mathbf{T}

$$\mathbf{H} = - \frac{1}{\beta} \ln \mathbf{T}. \quad (3)$$

The ground state energy for the quantum system described by Hamiltonian (3) is equal to the free energy (2),

$$e_0 = \lim_{\beta_q \rightarrow \infty} \frac{\text{Tr} (e^{-\beta_q \mathbf{H}})}{\text{Tr} e^{-\beta_q \mathbf{H}}} = - \frac{1}{\beta} \lim_{\beta_q \rightarrow \infty} \frac{\text{Tr} \left(\mathbf{T}^{\frac{\beta_q}{\beta}} \ln \mathbf{T} \right)}{\text{Tr} \mathbf{T}^{\frac{\beta_q}{\beta}}} = - \frac{1}{\beta} \ln \lambda_{\max} = f. \quad (4)$$

Here β_q is the inverse temperature of quantum system (3) and it must be distinguished from the inverse temperature of classical system (1) β . We will elaborate approximate method which allow to find Hamiltonian (3) for $2D$ classical systems (1) at low temperatures or with strong interaction in one spatial dimension.

The paper is organized as following. In the next (second) section the strong-anisotropy and the strong-interaction approximations in general for a $2D$ classical system are presented. In the third and fourth sections the methods are applied for

the 2D Ising model and the six-vertex model respectively. Quantum Hamiltonians are obtained there. Finally, in section five we compare the approximations by calculating critical temperatures for classical models. Important calculations which we use for the strong-interaction approximation are collected in Appendix.

2 Approximations for transfer matrix

In general, for an arbitrary 2D classical system with the total energy (1) it is impossible to find the transfer matrix logarithm and write down the Hamiltonian (3) explicitly. In this section we consider two approximate methods used to find the quantum system Hamiltonian (3). Both of them are based on division of energy E (1) into two parts, E_x which depends on configuration ξ of single row only and E_y which depends on two configurations ξ and ξ' of successive rows. There are many possible divisions, but we choose the division symmetric with respect to two neighboring rows

$$E(\xi, \xi') = \frac{E_x(\xi)}{2} + E_y(\xi, \xi') + \frac{E_x(\xi')}{2}. \quad (5)$$

Here E_x is the energy of a single row with configuration ξ and $E_y(\xi, \xi')$ is the energy of the interaction between two rows with configurations ξ and ξ' . Division (5) allows us to write $T_{\xi, \xi'} = T_{\xi}^x T_{\xi, \xi'}^y T_{\xi'}^x$, where $T_{\xi}^x = e^{-\beta \frac{E_x(\xi)}{2}}$, $T_{\xi, \xi'}^y = e^{-\beta E_y(\xi, \xi')}$. Accordingly we can rewrite T as a matrix product, $T = T^x T^y T^x$, and \mathbf{T} as an operator product,

$$\mathbf{T} = \sum_{\xi, \xi'=1}^{\mathbb{L}} |\xi\rangle T_{\xi, \xi'} \langle \xi'| = \mathbf{T}^x \mathbf{T}^y \mathbf{T}^x, \quad (6)$$

where $\mathbf{T}^x = \sum_{\xi=1}^{\mathbb{L}} |\xi\rangle T_{\xi}^x \langle \xi|$ is the diagonal operator and $\mathbf{T}^y = \sum_{\xi, \xi'=1}^{\mathbb{L}} |\xi\rangle T_{\xi, \xi'}^y \langle \xi'|$ is the off-diagonal operator.

For further progress we have to rewrite operator \mathbf{T}^y in a slightly different form. It is convenient to assume $E_y(\xi, \xi) = 0$ (that can be always achieved by including nonzero value $E_y(\xi, \xi)$ into $E_x(\xi)$ or by shifting all energies by a constant value). That makes diagonal elements of T^y equal to unity, $T_{\xi, \xi}^y = 1$. Among all off-diagonal elements of \mathbf{T}^y we distinguish the elements which are proportional to some parameter t_y (which we will demand later to be small) and collect these elements in operator \mathbf{Y}

$$\mathbf{Y}_{\xi, \xi'} = \begin{cases} T_{\xi, \xi'}^y, & \text{if } T_{\xi, \xi'}^y \propto t_y \\ 0, & \text{in other cases.} \end{cases} \quad (7)$$

All another off-diagonal elements which are $\mathcal{O}(t_y^2)$ we collect in operator \mathbf{Y}' . Summing up

$$\mathbf{T} = e^{t_x \mathbf{X}} (\mathbf{1} + t_y \mathbf{Y} + \mathcal{O}(t_y^2) \mathbf{Y}') e^{t_x \mathbf{X}}, \quad (8)$$

where diagonal part \mathbf{T}^x is rewritten in the form $\mathbf{T}^x = e^{t_x \mathbf{X}}$. Until now no approximation has been made. We only assume that the parameters t_x and t_y exist. For

both approaches, i.e., strong-anisotropy limit and strong-interaction limit it will be later required that t_y is small and we will neglect the terms $\mathcal{O}(t_y^2)$. For each specific model the individual operator \mathbf{Y} should be constructed bearing this requirement in mind.

2.1 The strong-anisotropy limit can be introduced by neglecting the terms $\mathcal{O}(t_y^2)$ and by a naive assumption - commutativity of \mathbf{X} and \mathbf{Y} in the expression for \mathbf{T} :

$$\mathbf{T} = e^{t_x \mathbf{X}} (\mathbf{1} + t_y \mathbf{Y} + \mathcal{O}(t_y^2) \mathbf{Y}') e^{t_x \mathbf{X}} = e^{t_x \mathbf{X}} e^{t_y \mathbf{Y}} e^{t_x \mathbf{X}} + \mathcal{O}(t_y^2) \underset{[\mathbf{X}, \mathbf{Y}] = 0}{\approx} e^{2t_x \mathbf{X} + t_y \mathbf{Y}}. \quad (9)$$

More precisely, the last approximation in (9) can be obtained by series expansions of the exponents and neglecting the terms $\mathcal{O}(t_x^2)$ and $\mathcal{O}(t_y^2)$,

$$\mathbf{T} = e^{t_x \mathbf{X}} (\mathbf{1} + t_y \mathbf{Y} + \mathcal{O}(t_y^2) \mathbf{Y}') e^{t_x \mathbf{X}} = e^{2t_x \mathbf{X} + t_y \mathbf{Y}} + \mathcal{O}(t_x^2) + \mathcal{O}(t_y^2). \quad (10)$$

Finally, the Hamiltonian in the strong-anisotropy limit has the form

$$\mathbf{H}_{sal} = -\frac{1}{\beta} \ln \mathbf{T} = -\frac{2t_x}{\beta} \mathbf{X} - \frac{t_y}{\beta} \mathbf{Y}. \quad (11)$$

For the strong-anisotropy approximation it is often assumed that $t_x \propto t_y$ and $\mathcal{O}(t_y^2) = 0$. This conditions impose some relations on the classical system parameters (1). Assumption $t_x \propto t_y$ is good to explain why approximation is called *strong-anisotropy* limit. If we consider the simplest case for model (1) with all in-row energies proportional to e_x , $E_x(\xi) \propto e_x$, and all inter-row interaction energies for different configurations not smaller than e_y , $E_y(\xi, \xi') \propto e_y$, $\xi \neq \xi'$, then the condition $t_x \propto t_y$ reads

$$\beta e_x \propto e^{-\beta e_y}. \quad (12)$$

By demanding $e^{-\beta e_y}$ to be small we suppose e_y to be large and e_x to be small. It should be noted that by demanding $e^{-\beta e_y}$ to be small we also assume that $e_y > 0$. Conditions $t_x \propto t_y$, $\mathcal{O}(t_y^2) = 0$ is weakest from a set of assumptions $t_x \in \mathcal{O}(t_y)$, $\mathcal{O}(t_y^2) = 0$ which allow to apply strong-anisotropy approach.

2.2 The strong-interaction limit, in contrast to the strong-anisotropy limit, demands only t_y to be small and does not put any restrictions on t_x . We can rewrite the expression for transfer matrix (8) in the form

$$\mathbf{T} = e^{t_x \mathbf{X}} (\mathbf{1} + t_y \mathbf{Y} + \mathcal{O}(t_y^2)) e^{t_x \mathbf{X}} = e^{2t_x \mathbf{X} + t_y (\mathbf{Y} + \mathbf{Z})} + \mathcal{O}(t_y^2) \quad (13)$$

where \mathbf{Z} is an unknown operator defined by the function $\mathbf{Z} = \mathbf{Z}(t_x \mathbf{X}, \mathbf{Y})$. The function \mathbf{Z} can be expand as series in the nested commutators of operators \mathbf{X} and \mathbf{Y} (see Appendix). If a general nested commutator $[\mathbf{X}, \mathbf{Y}]_n = [\mathbf{X}, [\mathbf{X}, \mathbf{Y}]_{n-1}]$ can be presented explicitly via some operators $\mathbf{L}, \mathbf{z}, \mathbf{R}$ (see A10), all calculations can be performed to the very end and \mathbf{Z} can be found explicitly. Thus, the Hamiltonian in the strong-interaction limit takes the form

$$\mathbf{H}_{sil} = -\frac{2t_x}{\beta} \mathbf{X} - \frac{t_y}{\beta} \mathbf{Y} - \frac{t_y}{\beta} \sum_{q=1}^Q \sum_{l=0}^{P-1} \mathbf{L}_{l,q} \mathcal{A}_{P,l}(t_x \mathbf{z}_{l,q}) \mathbf{R}_{l,q}, \quad (14)$$

where $\mathcal{A}_{P,l}(x) = \frac{1}{P} \sum_{p=0}^{P-1} e^{-\frac{2i\pi pl}{P}} \mathcal{A}\left(xe^{\frac{i\pi p}{P}}\right)$, $\mathcal{A}(x) = \frac{x}{\sinh x} - 1$. The strong-anisotropy limit (11) follows from the strong-interaction limit (14) as expected. If we assume $t_x \propto t_y$ and neglect the terms $\mathcal{O}(t_y^2)$ the last term in (14) vanishes since $\mathcal{A}_{P,l}(x) \in \mathcal{O}(x^2)$.

3 Two-dimensional Ising model

In this section both approximations will be used for the $2D$ classical Ising model in order to find the corresponding $1D$ quantum system. The classical $2D$ Ising model is described by the Hamiltonian

$$E = - \sum_{m=1}^M \sum_{n=1}^N J_x \sigma_{m,n} \sigma_{m,n+1} + J_y \sigma_{m,n} \sigma_{m+1,n}, \quad (15)$$

where $\sigma_{m,n}$ assumes two values $\pm \frac{1}{2}$. A configuration in each row m is defined by a set of variables $\{\sigma_{m,n}\}_{n=1}^N$ and it takes $\mathbb{L} = 2^N$ possible values. We will denote the configurations on two successive rows by $\{\sigma_n\}$ and $\{\sigma'_n\}$ and use the following notation

$$\begin{aligned} E_x(\{\sigma_n\}) &= - \sum_{n=1}^N J_x \sigma_n \sigma_{n+1}, \\ E_y(\{\sigma_n\}, \{\sigma'_n\}) &= - \sum_{n=1}^N J_y \sigma_n \sigma'_n. \end{aligned} \quad (16)$$

The Hilbert space can be spanned by the basis $\{|\xi\rangle\}_{\xi=1,2^N} = \{\otimes_{n=1}^N |\sigma_n\rangle\}_{\sigma_1 \dots \sigma_N = \pm \frac{1}{2}}$.

We can establish a correspondence between the states $|\sigma\rangle$, $\sigma = \pm \frac{1}{2}$ and the eigenvectors of the spin operator $\mathbf{s}^z = \sum_{\sigma} |\sigma\rangle \sigma \langle \sigma|$. Here and further on \sum_{σ} denotes $\sum_{\sigma = \pm \frac{1}{2}}$. The diagonal part of the transfer matrix \mathbf{X} has the form

$$\begin{aligned} \mathbf{T}^x &= \sum_{\sigma_1} \dots \sum_{\sigma_N} \bigotimes_{n=1}^N |\sigma_n\rangle e^{\beta \frac{J_x}{2} \sum_{j=1}^N \sigma_j \sigma_{j+1}} \bigotimes_{n=1}^N \langle \sigma_n| \\ &= e^{\beta \frac{J_x}{2} \sum_{j=1}^N \sum_{\sigma_1} \dots \sum_{\sigma_N} \bigotimes_{n=1}^N |\sigma_n\rangle \sigma_j \sigma_{j+1} \bigotimes_{n=1}^N \langle \sigma_n|} e^{\beta \frac{J_x}{2} \sum_{j=1}^N s_j^z s_{j+1}^z}, \end{aligned} \quad (17)$$

where we omitted the direct products of the identity operators. The off-diagonal operator \mathbf{T}^y takes the form

$$\mathbf{T}^y = \sum_{\sigma_1} \dots \sum_{\sigma_N} \sum_{\sigma'_1} \dots \sum_{\sigma'_N} \bigotimes_{n=1}^N |\sigma_n\rangle e^{\beta J_y \sum_{j=1}^N \sigma_j \sigma'_j} \bigotimes_{n=1}^N \langle \sigma'_n|. \quad (18)$$

In contrast to the case of diagonal operator \mathbf{T}^x , Eq. (17), we can not bring the direct products under the exponent. To construct the operator \mathbf{Y} , we classify all matrix elements $\mathbf{T}^y_{\{\sigma_n\}, \{\sigma'_n\}}$ into three groups depending upon a number of different variables in $\{\sigma_n\}$ and $\{\sigma'_n\}$. In the first group we collect all elements for which the

rows $\{\sigma_n\}, \{\sigma'_n\}$ are the same, i.e. the diagonal elements of \mathbf{T}^y . Diagonal elements are equal to $e^{\frac{N\beta J_y}{4}}$, as it was discussed above, we can obtain $\mathbf{T}^y_{\{\sigma_n\}, \{\sigma_n\}} = 1$ after shifting the energies by an appropriate quantity. All the matrix elements where the variables only on one cite differ, i.e., $\{\sigma_n = \sigma'_n\}_{n=1, \dots, N, n \neq j}, \sigma_j = -\sigma'_j$ will form the second group. In this case the interaction energy increases by $\frac{J_y}{2}$ and the matrix element of \mathbf{T}^y after energy shifting is equal to $e^{-\beta \frac{J_y}{2}}$. The third group consists of all other matrix elements for the states $\{\sigma_n\}, \{\sigma'_n\}$ which are different on two or more sites. These elements are equal to $\left(e^{-\beta \frac{J_y}{2}}\right)^r$, where r is the number of sites with different variables σ_n, σ'_n . Now, it can be seen how the small parameter t_y should be set. If $t_y = e^{-\beta \frac{J_y}{2}}$ is small, we can construct an operator \mathbf{Y} which has all matrix elements equal to zero except the elements between the states which differ by one variable σ_j (second group). All other matrix elements (third group) which are of order $\mathcal{O}(t_y^2)$ may be included into a non-important operator \mathbf{Y}' . Formally it can be done by rewriting the sum (18) as follows:

$$\begin{aligned}
\mathbf{T}^y &= \sum_{\sigma_1} \dots \sum_{\sigma_N} \bigotimes_{n=1}^N |\sigma_n\rangle \bigotimes_{n=1}^N \langle \sigma_n| \\
&+ t_y \sum_{j=1}^N \sum_{\sigma_1} \dots \sum_{\sigma_N} \bigotimes_{n=1}^{j-1} |\sigma_n\rangle \otimes |\sigma_j\rangle \bigotimes_{n=j+1}^N |\sigma_n\rangle \bigotimes_{n=1}^{j-1} \langle \sigma_n| \otimes \langle -\sigma_j| \bigotimes_{n=j+1}^N \langle \sigma_n| \\
&+ \sum_{\text{all other configurations}} e^{-\beta E_y(\{\sigma\}, \{\sigma'\})} \bigotimes_{n=1}^N |\sigma_n\rangle \bigotimes_{n=1}^N \langle \sigma'_n| \\
&= \mathbf{1} + t_y \mathbf{Y} + \mathcal{O}(t_y^2) \mathbf{Y}'.
\end{aligned} \tag{19}$$

(We notice that in order to calculate higher approximations with respect to t_y operator \mathbf{Y}' should be presented as series with respect to t_y^r , $r \geq 2$. The term proportional to t_y^r contains $\binom{N}{r}$ matrix elements of \mathbf{T}^y). The operator \mathbf{Y} can be easily identified in terms of spin operators,

$$\begin{aligned}
\mathbf{Y} &= \sum_{j=1}^N \sum_{\sigma_1} \dots \sum_{\sigma_N} \bigotimes_{n=1}^{j-1} |\sigma_n\rangle \otimes |\sigma_j\rangle \bigotimes_{n=j+1}^N |\sigma_n\rangle \bigotimes_{n=1}^{j-1} \langle \sigma_n| \otimes \langle -\sigma_j| \bigotimes_{n=j+1}^N \langle \sigma_n| \\
&= 2 \sum_{j=1}^N \mathbf{s}_j^x,
\end{aligned} \tag{20}$$

where $\mathbf{s}^x = \frac{1}{2} \sum_{\sigma} |\sigma\rangle \langle -\sigma|$. From Eqs. (17) and (20) we see that $t_x, t_y, \mathbf{X}, \mathbf{Y}$ can be written as

$$t_x = \frac{\beta J_x}{2}, \quad t_y = e^{-\beta \frac{J_y}{2}}, \quad \mathbf{X} = \sum_{j=1}^N \mathbf{s}_j^z \mathbf{s}_{j+1}^z, \quad \mathbf{Y} = \sum_{j=1}^N 2\mathbf{s}_j^x. \tag{21}$$

In the strong-anisotropy limit we obtain the Hamiltonian of the quantum Ising chain in transverse field

$$\mathbf{H}_{sal} = -\frac{2}{\beta}t_x\mathbf{X} - \frac{t_y}{\beta}\mathbf{Y} = -\sum_{j=1}^N J_x \mathbf{s}_j^z \mathbf{s}_{j+1}^z + \frac{2}{\beta}e^{-\beta\frac{J_y}{2}}\mathbf{s}_j^x. \quad (22)$$

To obtain the Hamiltonian in the strong-interaction limit (i.e., t_y is small) we have to consider the commutators $[\mathbf{X}, \mathbf{Y}]_{2k}$

$$[\mathbf{X}, \mathbf{Y}]_{2k} = 2 \sum_{j=1}^N \mathbf{s}_j^x \left(\frac{1}{2} + 2\mathbf{s}_{j-1}^z \mathbf{s}_{j+1}^z \right), \quad k \geq 1. \quad (23)$$

Comparing Eqs. (23) and (A10) we find that we have $P, Q = 1$, $l = 0$, $\mathbf{L}_{0,1} = \sum_j \mathbf{s}_j^x (1 + 4\mathbf{s}_{j-1}^z \mathbf{s}_{j+1}^z)$, $\mathbf{z}_{0,1} = 1$, $\mathbf{R}_{0,1} = 1$ and only one function $\mathcal{A}_{1,0}(t_x) = \mathcal{A}(t_x)$ in (A12) have to be calculated. Therefore, from (14)

$$\mathbf{H}_{sil} = -\sum_{j=1}^N J_x \mathbf{s}_j^z \mathbf{s}_{j+1}^z + \frac{2}{\beta}e^{-\beta\frac{J_y}{2}}\mathbf{s}_j^x \left(1 + \frac{1}{2}\mathcal{A}\left(\frac{\beta J_x}{2}\right) + 2\mathcal{A}\left(\frac{\beta J_x}{2}\right)\mathbf{s}_{j-1}^z \mathbf{s}_{j+1}^z \right). \quad (24)$$

In comparison to (22) the strong-interaction approximation (24) implies renormalized transverse field $\left[1 \rightarrow 1 + \frac{1}{2}\mathcal{A}\left(\frac{\beta J_x}{2}\right)\right]$ and additional three-spin interactions of $\mathbf{s}_{j-1}^z \mathbf{s}_j^x \mathbf{s}_{j+1}^z$ type.

4 Six-vertex model

In this section we will find the quantum Hamiltonian in two approximations for the six-vertex model [10]. The model consists of arrows on a two-dimensional square lattice. Arrows are associated to each link between the nearest lattice sites. The total energy is the sum over the lattice vertex energies. Each vertex energy depends on configurations of four neighboring arrows. Moreover, only six of such configurations are allowed. In Fig.1 the vertex configurations and the appropriate energies are shown.

Not all possible arrow configurations are allowed: for each vertex the arrow configuration must be one of a set depicted in Fig.1 by bold solid arrows. In this paper we will consider a lattice "rotated" by $\frac{\pi}{4}$ with periodic boundary conditions imposed. The total energy has the form

$$E = \sum_{m=1}^M \sum_{n=1}^N {}^{odd} E(\sigma_{m,n}, \sigma_{m,n+1}, \mu_{m,n}, \mu_{m,n+1}) + \sum_{n=1}^N {}^{even} E(\mu_{m,n}, \mu_{m,n+1}, \sigma_{m+1,n}, \sigma_{m+1,n+1}), \quad (25)$$

where the sum runs over all vertices. Arrows directions are encoded by variables σ and μ . We will denote the variables in the two nearest rows m and $m+1$ by

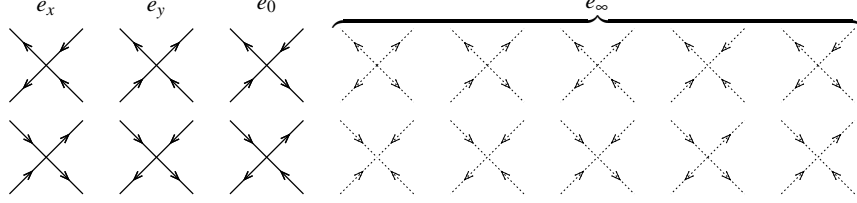


Fig. 1 All possible configurations of rows at a single vertex. The six-vertex model admits only the first six configurations (solid arrows) with finite energy. All configurations prohibited for six-vertex model (dotted arrows) have the energies that will be tend to infinity and the Boltzmann weights become zero.

$\{\sigma\}, \{\mu\}$ and $\{\sigma'\}, \{\mu'\}$ (see Fig. 2). The local energy for each vertex has the form

$$E(\sigma_1, \sigma_2, \mu_1, \mu_2) = e_0 \delta_{\sigma_1}^{\sigma_2} \delta_{\sigma_1 \mu_1} \delta_{\sigma_2 \mu_2} + e_x \delta_{\sigma_1 \sigma_2} \delta_{\sigma_1 \mu_1} \delta_{\sigma_2 \mu_2} + e_y \delta_{\sigma_1}^{\sigma_2} \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2}^{\mu_2} + e_\infty \delta_{\sigma_1 \sigma_2} (\delta_{\sigma_1 \mu_1} \delta_{\sigma_2}^{\mu_2} + \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2 \mu_2} + \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2}^{\mu_2}) + e_\infty \delta_{\sigma_1}^{\sigma_2} (\delta_{\sigma_1 \mu_1} \delta_{\sigma_2}^{\mu_2} + \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2}^{\mu_2}), \quad (26)$$

where δ_{ab} is the Kronecker symbol, $\delta_a^b = 1 - \delta_{ab}$. All forbidden configurations appear in (26) with energies e_∞ which will be later sent to infinity and the Boltzmann weights for the forbidden configurations will go to zero. By using the local energy symmetry $E(\mu_1, \mu_2, \sigma_1, \sigma_2) = E(\sigma_1, \sigma_2, \mu_1, \mu_2)$ and the trivial identity $e_x \delta_{\sigma_1 \sigma_2} \delta_{\sigma_1 \mu_1} \delta_{\sigma_2 \mu_2} \equiv e_x \delta_{\sigma_1 \sigma_2} (1 - \delta_{\sigma_1 \mu_1} \delta_{\sigma_2}^{\mu_2} - \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2 \mu_2} - \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2}^{\mu_2})$ we can divide energy in two parts

$$\begin{aligned} E_x(\sigma_1, \sigma_2, \mu_1, \mu_2) &= e_x \delta_{\sigma_1, \sigma_2}, \\ E_y(\sigma_1, \sigma_2, \mu_1, \mu_2) &= e_0 \delta_{\sigma_1}^{\sigma_2} \delta_{\sigma_1 \mu_1} \delta_{\sigma_2 \mu_2} + 0 \cdot \delta_{\sigma_1 \sigma_2} \delta_{\sigma_1 \mu_1} \delta_{\sigma_2 \mu_2} + e_y \delta_{\sigma_1}^{\sigma_2} \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2}^{\mu_2} \\ &\quad + (e_\infty - e_x) \delta_{\sigma_1 \sigma_2} (\delta_{\sigma_1 \mu_1} \delta_{\sigma_2}^{\mu_2} + \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2 \mu_2} + \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2}^{\mu_2}) \\ &\quad + e_\infty \delta_{\sigma_1}^{\sigma_2} (\delta_{\sigma_1 \mu_1} \delta_{\sigma_2}^{\mu_2} + \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2}^{\mu_2}). \end{aligned} \quad (27)$$

A reason for introducing the term proportional to zero in (27) will be seen later. Now, following our scheme we can write down the two components of the transfer matrix

$$T_{\{\sigma, \mu\}, \{\sigma', \mu'\}}^x = \prod_{n=1}^N \delta_{\sigma_n \sigma'_n} \delta_{\mu_n \mu'_n} e^{-\beta \frac{e_x}{2} \sum_{n=1}^N \delta_{\sigma_n \sigma_{n+1}}}, \quad (28)$$

$$T_{\{\sigma, \mu\}, \{\sigma', \mu'\}}^y = \prod_{n=1}^N \text{odd} e^{-\beta E_y(\sigma_n, \sigma_{n+1}, \mu_n, \mu_{n+1})} \prod_{n=1}^N \text{even} e^{-\beta E_y(\sigma'_n, \sigma'_{n+1}, \mu_n, \mu_{n+1})}. \quad (29)$$

Let us consider $e^{-\beta E_y(\sigma_1, \sigma_2, \mu_1, \mu_2)}$ in more details. As it can be seen in (27), E_y depends on four variables $\sigma_1, \sigma_2, \mu_1, \mu_2$. Each of these variables can take two values $\pm \frac{1}{2}$, and therefore, set of four variables can take 2^4 different sets of values. For each set of values of $\sigma_1, \sigma_2, \mu_1, \mu_2$ one and only one term in (27) is nonzero, and

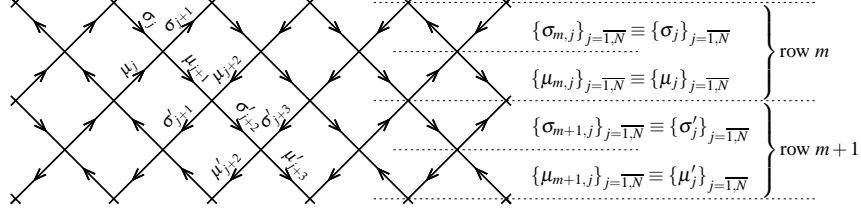


Fig. 2 Some permitted configurations for the six-vertex model. We show two neighbor rows to explain a definition of variables σ and μ .

therefore we can write

$$\begin{aligned} e^{-\beta E_y(\sigma_1, \sigma_2, \mu_1, \mu_2)} = & e^{-\beta e_0} \delta_{\sigma_1}^{\sigma_2} \delta_{\sigma_1 \mu_1} \delta_{\sigma_2 \mu_2} + e^{-\beta 0} \delta_{\sigma_1 \sigma_2} \delta_{\sigma_1 \mu_1} \delta_{\sigma_2 \mu_2} + e^{-\beta e_y} \delta_{\sigma_1}^{\sigma_2} \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2}^{\mu_2} \\ & + e^{-\beta e_\infty} e^{\beta e_x} \delta_{\sigma_1 \sigma_2} (\delta_{\sigma_1 \mu_1} \delta_{\sigma_2}^{\mu_2} + \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2 \mu_2} + \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2}^{\mu_2}) \\ & + e^{-\beta e_\infty} \delta_{\sigma_1}^{\sigma_2} (\delta_{\sigma_1 \mu_1} \delta_{\sigma_2}^{\mu_2} + \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2 \mu_2}). \end{aligned} \quad (30)$$

We are free to redefine the energies, $e_x \rightarrow e_x - e_0$, $e_y \rightarrow e_y - e_0$, $e_0 \rightarrow 0$, and to send the energies of forbidden configurations to infinity, $e^{-\beta e_\infty} \rightarrow 0$. By denoting $t_y = e^{-\beta e_y}$ we get

$$e^{-\beta E_y(\sigma_1, \sigma_2, \mu_1, \mu_2)} = \delta_{\sigma_1 \mu_1} \delta_{\sigma_2 \mu_2} + t_y \delta_{\sigma_1}^{\sigma_2} \delta_{\sigma_1}^{\mu_1} \delta_{\sigma_2}^{\mu_2}. \quad (31)$$

Now we have

$$\begin{aligned} T_{\{\sigma, \mu\}, \{\sigma', \mu'\}}^y &= \prod_{n=1}^N \text{odd} [\delta_{\sigma_n \mu_n} \delta_{\sigma_{n+1} \mu_{n+1}} + t_y \delta_{\sigma_n}^{\sigma_{n+1}} \delta_{\sigma_n}^{\mu_n} \delta_{\sigma_{n+1} \mu_{n+1}}] \\ &\times \prod_{n=1}^N \text{even} [\delta_{\sigma'_n \mu_n} \delta_{\sigma'_{n+1} \mu_{n+1}} + t_y \delta_{\sigma'_n}^{\sigma'_{n+1}} \delta_{\sigma'_n}^{\mu_n} \delta_{\sigma'_{n+1} \mu_{n+1}}] \\ &= \prod_{n=1}^N \delta_{\sigma_n \mu_n} \delta_{\sigma'_n \mu_n} + t_y \sum_{j=1}^N \prod_{n=1}^N \prod_{n \neq j+1}^{n \neq j} \delta_{\sigma_n, \mu_n} \delta_{\sigma'_n, \mu_n} \\ &\times \begin{cases} \delta_{\sigma_j}^{\sigma_{j+1}} \delta_{\sigma_j}^{\mu_j} \delta_{\sigma_{j+1}}^{\mu_{j+1}} \delta_{\sigma'_j \mu_j} \delta_{\sigma'_{j+1} \mu_{j+1}}, & \text{if } j \text{ odd} \\ \delta_{\sigma'_{j+1}}^{\sigma'_j} \delta_{\sigma'_j}^{\mu_j} \delta_{\sigma'_{j+1}}^{\mu_{j+1}} \delta_{\sigma_j \mu_j} \delta_{\sigma_{j+1} \mu_{j+1}}, & \text{if } j \text{ even} \end{cases} + \mathcal{O}(t_y^2). \end{aligned} \quad (32)$$

Using the identities $\delta_{ab} \delta_{bc} \equiv \delta_{ab} \delta_{ac}$, $\delta_{ab} \delta_b^c \equiv \delta_{ab} \delta_a^c$ we can rewrite T^y in the form

$$\begin{aligned} T_{\{\sigma, \mu\}, \{\sigma', \mu'\}}^y &= \prod_{n=1}^N \delta_{\sigma_n, \sigma'_n} \delta_{\sigma_n, \mu_n} \\ &+ t_y \sum_{j=1}^N \prod_{n=1}^N \prod_{n \neq j+1}^{n \neq j} \delta_{\sigma_n \sigma'_n} \delta_{\sigma_n \mu_n} \delta_{\sigma_j}^{\sigma_{j+1}} \delta_{\sigma'_j}^{\sigma'_{j+1}} \times \begin{cases} \delta_{\sigma'_j \mu_j} \delta_{\sigma'_{j+1} \mu_{j+1}}, & \text{if } j \text{ odd} \\ \delta_{\sigma_j \mu_j} \delta_{\sigma_{j+1} \mu_{j+1}}, & \text{if } j \text{ even} \end{cases} + \mathcal{O}(t_y^2). \end{aligned} \quad (33)$$

The matrices T , T^x and T^y are of the size $2^{2N} \times 2^{2N}$ and the quantum Hamiltonian $\mathbf{H} = -\frac{1}{\beta} \ln \mathbf{T}$ should be defined in the 2^{2N} -dimensional Hilbert space. We can show that in the particular case (29) the space dimension can be reduced to 2^N . Let us recall that we are looking for thermodynamic properties of the system described by the partition function $\text{Tr } T^M$ (2). Matrix (29) can be represented in the form $T^y = T_1^y + t_y T_2^y + \mathcal{O}(t_y^2)$ and by using the trace properties we may write

$$\begin{aligned} \text{Tr } T^M &= \text{Tr} \left(T^x (T_1^y + t_y T_2^y + \mathcal{O}(t_y^2)) T^x \right)^M \\ &= \text{Tr} \left[((T^x)^2 T_1^y)^M + M t_y ((T^x)^2 T_1^y)^{M-1} (T^x)^2 T_2^y \right] + \mathcal{O}(t_y^2). \end{aligned} \quad (34)$$

The trace over space $\{\sigma, \mu\}$ can be splitted into two partial traces over $\{\sigma\}$ and over $\{\mu\}$. From Eqs. (28), (29) one can easily write down the matrix elements for $((T^x)^2 T_1^y)^M$ and $M t_y ((T^x)^2 T_1^y)^{M-1} T_2^y$ and perform the partial trace $\text{Tr}_{\{\mu\}}$. The result is as follows

$$\begin{aligned} \text{Tr } T^M &= \text{Tr}_{\{\sigma\}} \text{Tr}_{\{\mu\}} \left[((T^x)^2 T_1^y)^M + M t_y ((T^x)^2 T_1^y)^{M-1} (T^x)^2 T_2^y \right] + \mathcal{O}(t_y^2) \\ &= \text{Tr}_{\{\sigma\}} e^{-\beta M e_x \sum_j \delta \sigma_j \sigma_{j+1}} \left[\prod_{n=1}^N \delta_{\sigma_n \sigma'_n} + M t_y \sum_{j=1}^N \prod_{n=1}^N \delta_{\sigma_n \sigma'_n} \delta_{\sigma_j \sigma'_{j+1}} \delta_{\sigma'_j \sigma'_{j+1}} \right] \\ &\quad + \mathcal{O}(t_y^2). \end{aligned} \quad (35)$$

The latter expression in Eq. (35) is $\text{Tr}_{\{\sigma\}} (\tilde{T}^x \tilde{T}^y \tilde{T}^x)^M + \mathcal{O}(t_y^2)$, where the matrices \tilde{T}^x , \tilde{T}^y are defined by their elements

$$\begin{aligned} \tilde{T}_{\{\sigma\}, \{\sigma'\}}^x &= \prod_{n=1}^N \delta_{\sigma_n \sigma'_n} e^{-\beta \frac{e_x}{2} \sum_{j=1}^N \delta \sigma_j \sigma_{j+1}}, \\ \tilde{T}_{\{\sigma\}, \{\sigma'\}}^y &= \prod_{n=1}^N \delta_{\sigma_n \sigma'_n} + t_y \sum_{j=1}^N \prod_{n=1}^N \delta_{\sigma_n \sigma'_n} \delta_{\sigma_j \sigma'_{j+1}} \delta_{\sigma'_j \sigma'_{j+1}} + \mathcal{O}(t_y^2). \end{aligned} \quad (36)$$

We can look for the ground-state energy of Hamiltonian $\tilde{\mathbf{H}}$, which acts in the 2^N -dimensional Hilbert space $\{\sigma\}$ and is defined by the logarithm of the matrix $\tilde{T} = \tilde{T}^x \tilde{T}^y \tilde{T}^x$. The reason that we can perform space dimension reduction is as follows: For any two states $\{\sigma\}$, $\{\sigma'\}$, which differ by only one σ_j , there is only one configuration $\{\mu\}$, for which the matrix element of T remains non-vanishing in the limit $\mathcal{O}(t_y^2) = 0$. From here we will omit the tilde and use the notations T , T^x and T^y to denote the matrices that depend only on $\{\sigma\}$ variables. Matrices \tilde{T}^x , \tilde{T}^y (36) are represented in the form suitable for obtaining operators \mathbf{X} and \mathbf{Y} mentioned in (8).

$$\begin{aligned} \mathbf{X} &= \sum_{j=1}^N \sum_{\{\sigma\}} \bigotimes_{n=1}^N |\sigma_n\rangle \delta_{\sigma_j \sigma_{j+1}} \bigotimes_{n=1}^N \langle \sigma_n|, \\ \mathbf{Y} &= \sum_{j=1}^N \sum_{\{\sigma, \sigma'\}} \bigotimes_{n=1}^N |\sigma_n\rangle \prod_{n=1}^N \delta_{\sigma_n \sigma'_n} \delta_{\sigma_j \sigma'_{j+1}} \delta_{\sigma'_j \sigma'_{j+1}} \bigotimes_{n=1}^N \langle \sigma'_n|. \end{aligned} \quad (37)$$

If the states defined by $\{\sigma\}$ are considered as the spin states $|\frac{1}{2}\rangle = |\uparrow\rangle$, $|\frac{-1}{2}\rangle = |\downarrow\rangle$, we can immediately recognize the spin operators

$$\sum_{\sigma_j, \sigma_{j+1}} |\sigma_j\rangle \otimes |\sigma_{j+1}\rangle \delta_{\sigma_j \sigma_{j+1}} \langle \sigma_j| \otimes \langle \sigma_{j+1}| = 2(\mathbf{s}_j^z \mathbf{s}_{j+1}^z + 1/4) \quad (38)$$

$$\sum_{\sigma_j, \sigma_{j+1}} \sum_{\sigma'_j, \sigma'_{j+1}} |\sigma_j\rangle \otimes |\sigma_{j+1}\rangle \delta_{\sigma_j \sigma'_j}^{\sigma'_j} \delta_{\sigma_{j+1} \sigma'_{j+1}}^{\sigma'_{j+1}} \langle \sigma'_j| \otimes \langle \sigma'_{j+1}| = 2(\mathbf{s}_j^x \mathbf{s}_{j+1}^x + \mathbf{s}_j^y \mathbf{s}_{j+1}^y), \quad (39)$$

We have almost all that we need to construct the quantum Hamiltonian in both approximations. At last we have only to recall Eq. (28) to define t_x and collect essential variables and operators

$$t_x = -\frac{\beta e_x}{2}, \quad t_y = e^{-\beta e_y}, \quad \mathbf{X} = 2 \sum_{j=1}^N \mathbf{s}_j^z \mathbf{s}_{j+1}^z + 1/4, \quad \mathbf{Y} = 2 \sum_{j=1}^N \mathbf{s}_j^x \mathbf{s}_{j+1}^x + \mathbf{s}_j^y \mathbf{s}_{j+1}^y. \quad (40)$$

In the strong-anisotropy limit (11) we can immediately write the quantum Hamiltonian, which corresponds to the spin-1/2 XXZ chain

$$\mathbf{H}_{sal} = 2 \sum_{j=1}^N \left(e_x \mathbf{s}_j^z \mathbf{s}_{j+1}^z - \frac{1}{\beta} e^{-\beta e_y} (\mathbf{s}_j^x \mathbf{s}_{j+1}^x + \mathbf{s}_j^y \mathbf{s}_{j+1}^y) \right), \quad (41)$$

where we omit the insignificant constants. For the strong-interaction limit we have to calculate the commutator

$$[\mathbf{X}, \mathbf{Y}]_{2k} = 2^{2k} \sum_{j=1}^N \left(1 - 4 \mathbf{s}_{j-1}^z \mathbf{s}_{j+2}^z \right) (\mathbf{s}_j^x \mathbf{s}_{j+1}^x + \mathbf{s}_j^y \mathbf{s}_{j+1}^y). \quad (42)$$

This commutator has the form (A10) with $P, Q = 1, l = 0, \mathbf{L}_{0,1} = \sum_j (1 - 4 \mathbf{s}_{j-1}^z \mathbf{s}_{j+2}^z) \cdot (\mathbf{s}_j^x \mathbf{s}_{j+1}^x + \mathbf{s}_j^y \mathbf{s}_{j+1}^y)$, $\mathbf{z}_{0,1} = 2, \mathbf{R}_{0,1} = 1$. Since $P = 1$, we need calculate only one function in (A12) $\mathcal{A}_{0,1}(t_x) = \mathcal{A}(t_x) \equiv \mathcal{A}(-t_x)$, in order to construct operator \mathbf{Z} (A13). The Hamiltonian (14) becomes

$$\mathbf{H}_{sil} = 2 \sum_{j=1}^N e_x \mathbf{s}_j^z \mathbf{s}_{j+1}^z - \frac{1}{\beta} e^{-\beta e_y} \left(1 + \frac{\mathcal{A}(\beta e_x)}{2} (1 - 4 \mathbf{s}_{j-1}^z \mathbf{s}_{j+2}^z) \right) (\mathbf{s}_j^x \mathbf{s}_{j+1}^x + \mathbf{s}_j^y \mathbf{s}_{j+1}^y). \quad (43)$$

5 Conclusions

To compare the results provided by both approximations we will discuss the critical temperature for the Ising model which corresponds to quantum phase transition of the Hamiltonians obtained in the strong-anisotropy limit (22) and in the strong-interaction limit (24). Both quantum (22) and (24) models are particular cases of generalized spin- $\frac{1}{2}$ XY chain for which a critical point can be found by Jordan-Wigner and Bogolubov transformations [11]. Moreover, both quantum models

(22) and (24) have the critical point at the parameters which correspond to the same value for the critical temperature of the classical 2D Ising model

$$\frac{\beta_{sal} J_x}{4} \exp \frac{\beta_{sal} J_y}{2} = 1, \quad \beta_{sil} \equiv \beta_{sal}. \quad (44)$$

The fact that more accurate approximation, i.e. the strong-interaction approximation, does not give any improvement for the value of the critical temperature can be explained based on analysis of the well-known exact equation for the Ising model critical temperature,

$$\sinh \frac{\beta_{ex} J_x}{2} \sinh \frac{\beta_{ex} J_y}{2} = 1. \quad (45)$$

Eq. (45) can be rewritten in terms of t_x, t_y (21) as follows:

$$t_x = \operatorname{arsinh} \left(\frac{2t_y}{1+t_y^2} \right) = 2t_y + \mathcal{O}(t_y^3). \quad (46)$$

In the strong-interaction limit $\mathcal{O}(t_y^2) = 0$ and we have $t_x \propto t_y$ that is the requirement of the strong-anisotropy limit. The fact that both gave the same critical temperature can be explained in terms of interactions J_x, J_y and inverse temperature β . In space of parameters J_x, J_y, β the strong-interaction limit covers much wider region than the strong-anisotropy limit. But near to the surface of critical temperatures defined by Eq. (45) these regions coincide. More precisely intersections of two surfaces given by exact solutions to (45) and by the strong-interaction limit $e^{-\beta \frac{J_y}{2}} = 0$ is a line defined by the strong-anisotropy limit $e^{-\beta \frac{J_y}{2}} = 0, \frac{\beta J_x}{2} = 0$. This transparent geometrical interpretation unfortunately can not be simply depicted because all regions which are object of our interest are infinitely distant.

For the six-vertex model the critical temperature in the strong-anisotropy limit can be calculated from isotropy condition for the XXZ Hamiltonian, (41)

$$e^{-\beta_{ex} e_y} = \beta_{ex} e_x. \quad (47)$$

The critical temperature in the strong-interaction limit requires a study of the quantum spin chain described by Hamiltonian (43) for which no exact results are available. From the exact equation for the six-vertex model critical temperature,

$$e^{-\beta_{ex} e_x} + e^{-\beta_{ex} e_y} = 1, \quad (48)$$

we can draw conclusions similar to those derived for the Ising model. In fact, equation (48) can be rewritten (see (40)) as

$$t_x = \frac{\ln(1-t_y)}{2} = -\frac{t_y}{2} + \mathcal{O}(t_y^2), \quad (49)$$

wherefrom the equivalence of two approximations near the critical temperature may be expected. In Fig. 3 the results for critical temperature given by strong-anisotropy approximation and strong-interaction approximation are shown. We can see that the strong-anisotropy limit gives reasonable results even when the classical system become isotropic.

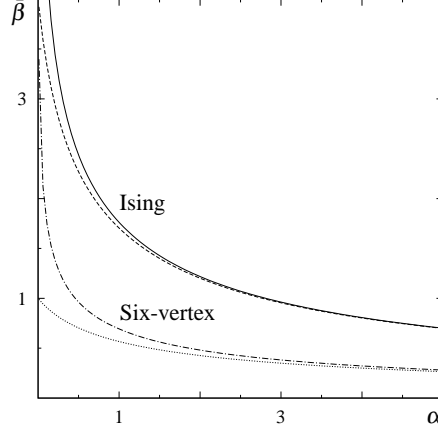


Fig. 3 Dependence of relative inverse critical temperature $\bar{\beta}$ on spatial anisotropy α . For the Ising model $\bar{\beta} = \beta J_x$, $\alpha = \frac{J_x}{J_y}$, solid line corresponds to exact solution (45), dashed line corresponds to the strong-anisotropy and strong-interaction limits (44). For the six-vertex model $\bar{\beta} = \beta e_x$, $\alpha = \frac{e_y}{e_x}$, dash-dotted line corresponds to exact solution (48), dotted line corresponds to the strong-anisotropy limit (47).

In spite the fact that the novel strong-interaction approximation does not improve the critical temperature of the considered classical 2D models, the strong-interaction approximation have several advantages. First, we can study isotropic classical systems. Second, we can apply strong-interaction approximation as low-temperature approximation for classical systems with arbitrary interactions. Indeed we can achieve $t_y \propto e^{-\beta e_y}$ to be small in two ways: by assuming strong interactions along one direction $e_y \rightarrow \infty$ or putting low temperature $\beta \rightarrow \infty$.

Appendix: Representation of exponent products via nested commutators

We will start from the expression for transfer matrix (13), where an unknown operator \mathbf{Z} was introduced $e^{t_x \mathbf{X}}(1 + t_y \mathbf{Y} + \mathcal{O}(t_y^2))e^{t_x \mathbf{X}} = e^{2t_x \mathbf{X} + t_y(\mathbf{Y} + \mathbf{Z})} + \mathcal{O}(t_y^2)$. By expanding the right-hand side of the latter equation in powers of t_y up to first order we derive for the linear over t_y terms the following result

$$e^{t_x \mathbf{X}} \mathbf{Y} e^{t_x \mathbf{X}} = \int_0^1 d\tau e^{2\tau t_x \mathbf{X}} (\mathbf{Y} + \mathbf{Z}) e^{-2\tau t_x \mathbf{X}} e^{2t_x \mathbf{X}}. \quad (\text{A1})$$

The left and the right multiplication by $e^{-t_x \mathbf{X}}$ and the substitution $\tau \rightarrow \frac{\tau+1}{2}$, $\int_0^1 d\tau \rightarrow \frac{1}{2} \int_{-1}^1 d\tau$ lead us to equation:

$$\mathbf{Y} = \frac{1}{2} \int_{-1}^1 e^{\tau t_x \mathbf{X}} (\mathbf{Y} + \mathbf{Z}) e^{-\tau t_x \mathbf{X}} d\tau = \frac{1}{2} \int_{-1}^1 \sum_{n=0}^{\infty} \frac{\tau^n t_x^n ([\mathbf{X}, \mathbf{Y}]_n + [\mathbf{X}, \mathbf{Z}]_n)}{n!} d\tau, \quad (\text{A2})$$

$$[\mathbf{A}, \mathbf{B}]_n = [\mathbf{A}, [\mathbf{A}, \mathbf{B}]_{n-1}], \quad [\mathbf{A}, \mathbf{B}]_0 = \mathbf{B}.$$

Now we can integrate over τ

$$\mathbf{Y} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{(n+1)!} t_x^n ([\mathbf{X}, \mathbf{Y}]_n + [\mathbf{X}, \mathbf{Z}]_n) = \sum_{n=0}^{\infty} t_x^{2n} \frac{[\mathbf{X}, \mathbf{Y}]_{2n} + [\mathbf{X}, \mathbf{Z}]_{2n}}{(2n+1)!}. \quad (\text{A3})$$

Let us presume that operator \mathbf{Z} can be expanded in terms of nested commutators with unknown coefficients A_k : $\mathbf{Z} = \sum_{k=1}^{\infty} \frac{A_k}{k!} t_x^k [\mathbf{X}, \mathbf{Y}]_k$. We can show that the function $Z(t_x \mathbf{X}, \mathbf{Y})$ is even with respect to the first argument,

$$\left. \begin{aligned} (e^{t_x \mathbf{X}} e^{t_y \mathbf{Y}} e^{t_x \mathbf{X}})^{-1} &= (e^{2t_x \mathbf{X} + t_y (\mathbf{Y} + Z(t_x \mathbf{X}, \mathbf{Y}))})^{-1} = e^{-2t_x \mathbf{X} - t_y (\mathbf{Y} + Z(t_x \mathbf{X}, \mathbf{Y}))} \\ (e^{t_x \mathbf{X}} e^{t_y \mathbf{Y}} e^{t_x \mathbf{X}})^{-1} &= e^{-t_x \mathbf{X}} e^{-t_y \mathbf{Y}} e^{-t_x \mathbf{X}} = e^{-2t_x \mathbf{X} - t_y (\mathbf{Y} + Z(-t_x \mathbf{X}, \mathbf{Y}))} \end{aligned} \right\} Z(t_x \mathbf{X}, \mathbf{Y}) = Z(-t_x \mathbf{X}, \mathbf{Y}), \quad (\text{A4})$$

that means that all odd A_{2k+1} are zero $A_{2k+1} \equiv 0$. Thus we will look for the following series for \mathbf{Z} :

$$\mathbf{Z} = \sum_{k=1}^{\infty} \frac{A_{2k}}{2k!} t_x^{2k} [\mathbf{X}, \mathbf{Y}]_{2k} \quad (\text{A5})$$

Substituting (A5) into (A3) and reordering the sum (Cauchy product) $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F(n, k) = \sum_{i=0}^{\infty} \sum_{j=0}^i F(j, i-j)$ we can write

$$\mathbf{Y} = \sum_{i=0}^{\infty} \frac{1}{(2i+1)!} t_x^{2i} [\mathbf{X}, \mathbf{Y}]_{2i} + \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} \frac{A_{2(i-j)}}{(2i-2j)!(2j+1)!} \right) t_x^{2i} [\mathbf{X}, \mathbf{Y}]_{2i}. \quad (\text{A6})$$

Equating the coefficients at equal powers of t_x we have the expressions for A_{2k}

$$\sum_{j=0}^{i-1} \frac{A_{2(i-j)}}{(2i-2j)!(2j+1)!} + \frac{1}{(2i+1)!} = 0. \quad (\text{A7})$$

From this equations we can write a recursive representation for A_k ,

$$A_{2i} = -\frac{1}{2i+1} \sum_{j=1}^i \binom{2i+1}{2j+1} A_{2(i-j)}, \quad i \geq 1 \quad A_0 = 1, \quad (\text{A8})$$

where $\binom{2i+1}{2j+1} = \frac{(2i+1)!}{(2j+1)!(2i-2j)!}$. It should be mentioned that for the Bernoulli numbers similar recursion representation exists,

$$B_{2i} = -\frac{1}{2i+1} \sum_{j=1}^i \binom{2i+1}{2j+1} B_{2(i-j)} + \frac{1}{2}, \quad i \geq 1 \quad B_0 = 1, B_1 = -\frac{1}{2}. \quad (\text{A9})$$

Our new task is to find the expansion (A5). Evidently it is impossible to do for arbitrary operators \mathbf{X} and \mathbf{Y} . We will consider only the case in which the commutator $[\mathbf{X}, \mathbf{Y}]_{2k}$ have following periodical (with period P) structure:

$$[\mathbf{X}, \mathbf{Y}]_{2Pr+2l} = \sum_{q=1}^Q \mathbf{L}_{l,q} (\mathbf{z}_{l,q})^{2Pr+2l} \mathbf{R}_{l,q}, \quad (\text{A10})$$

where $P, Q, r \in \mathbb{N}$, $l = \overline{0, P-1}$ and the operators $\mathbf{L}_{l,q} \equiv \mathbf{L}_{l+P,q}$, $\mathbf{R}_{l,q} \equiv \mathbf{R}_{l+P,q}$, $\mathbf{z}_{l,q} \equiv \mathbf{z}_{l+P,q}$. Formula (A10) links nested commutators of the orders $2Pr+2l$, $r = \overline{1, \infty}$ by Q rules. For each $l = \overline{0, P-1}$ this rules can be different. With increasing the commutator order by $2P$ the expression for commutator is multiplied by operators \mathbf{z} with constant sandwich multiplication by the operators \mathbf{L} , \mathbf{R} . If commutator $[\mathbf{X}, \mathbf{Y}]_{2Pr+2l}$ have the form (A10) starting from $r' > 1$, we can separate in (A5) the terms $k = \overline{1, r'-1}$ and perform all computations for the redefined operator

Z. Further progress is possible due to fact that an exponential generating function $\mathcal{A}(x)$ for the coefficients A_{2k} can be suggested

$$\mathcal{A}(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{A_{2k}}{2k!} x^{2k} \equiv \frac{x}{\sinh x} - 1. \quad (\text{A11})$$

From Eq. (A11) we can find a generating function for $\mathcal{A}_{P,l}(x)$,

$$\mathcal{A}_{P,l}(x) \stackrel{\text{def}}{=} \sum_{r=1}^{\infty} \frac{A_{2Pr+2l}}{(2Pr+2l)!} x^{2Pr+2l} = \frac{1}{P} \sum_{p=0}^{P-1} e^{-\frac{2i\pi pl}{P}} \mathcal{A}\left(xe^{\frac{i\pi p}{P}}\right) \quad (\text{A12})$$

Finally, the unknown operator **Z** takes the form

$$\mathbf{Z} = \sum_{q=1}^Q \sum_{l=0}^{P-1} \mathbf{L}_{l,q} \mathcal{A}_{P,l}(t_x \mathbf{z}_{l,q}) \mathbf{R}_{l,q}. \quad (\text{A13})$$

Having obtained this operator we can take logarithm of the transfer matrix (13) and obtain quantum Hamiltonian in the strong-interaction limit (14)

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